Metric Spaces and Topology
Lecture 6

Prop. If a subsequence of a Cauchy sequence converges to $x \in X$, then the whole sequence converges to x.

Proof. Let $(x_n) \subseteq X$ be a Cauchy sequence and (x_{n_k}) a subsequence that converges to some $x \in X$.

$$d(x_k, x) \overset{\Delta}{\leq} d(x_k, x_{n_k}) + d(x_{n_k}, x)$$
$$\leq \operatorname{diam}\{x_k, x_{k+1}, \dots, x_{n_k}, \dots\} + d(x_{n_k}, x)$$
$$\to 0 + 0 \quad \text{as } k \to \infty. \qquad \square$$

Complete metric spaces. A metric space (X,d) is called *complete* if every Cauchy sequence $(x_n) \subseteq X$ has a limit $x \in X$.

Easy examples.

- Discrete metric spaces, i.e. where $d(x,y) = 1$ for all $x \neq y$. These are complete since every Cauchy sequence is eventually constant.
- $[0,1)$ with the usual metric is not complete because $\left(1 - \frac{1}{n}\right)$ is Cauchy (because it converges within $[0,1]$) but doesn't converge in $[0,1)$.

- $\mathbb{Q}$ with the usual metric is not complete because $\exists (q_n) \subseteq \mathbb{Q}$ converging to $\sqrt{2}$ in $\mathbb{R}$, so (q_n) is Cauchy but doesn't converge within $\mathbb{Q}$.

  Fact. $\sqrt{2} \notin \mathbb{Q}$, more formally, the polynomial $x^2 - 2 = 0$ doesn't have a root in $\mathbb{Q}$ (hence $\mathbb{Q}$ is not algebraically closed.

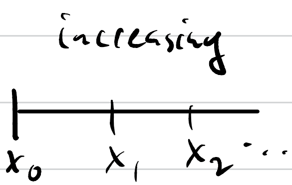
  Proof. Let $\frac{n}{m} = \sqrt{2}$ be the reduced form, then $n^2 = 2m^2$, so n is even, i.e. $n = 2\tilde{n}$, hence $2m^2 = 4\tilde{n}^2$, so $m^2 = 2\tilde{n}^2$, so m is even, so $m = 2\hat{m}$, so $\frac{n}{m}$ wasn't reduced. $\square$

The following illustrates how one may use completeness of a larger space to conclude something in the smaller space.

Prop In the metric space $\mathbb{Q}$ with the usual metric, bounded monotone sequences are Cauchy.

Proof 1 (using $\mathbb{R}$). We look at this sequence in $\mathbb{R}$. Then by the Monotone Convergence Theorem, this sequence converges in $\mathbb{R}$ and hence is Cauchy.

Proof 2 (without $\mathbb{R}$). Let $(x_n) \subseteq \mathbb{Q}$ be, say, increasing, and suppose it is not Cauchy. We'll show that it's not bounded.



Not Cauchy implies that $\exists\, \varepsilon > 0$ s.t. $\forall n$, $\operatorname{diam}(\{x_n, x_{n+1}, \dots\}) > \varepsilon$ (this is because $\operatorname{diam}(\{x_n, x_{n+1}, \dots\})$ is a decreasing sequence). Then $\exists\, x_{n_1}$ s.t. $d(x_0, x_{n_1}) > \varepsilon$. $\exists\, x_{n_2}$ s.t. $d(x_{n_1}, x_{n_2}) > \varepsilon$, ... thus, $d(x_0, x_{n_k}) \geq k \cdot \varepsilon$, so (x_n) is unbdd. $\square$

Characterization of completeness (in terms of nonempty intersection)

For a metric space (X, d), TFAE:

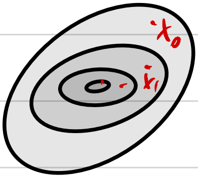
(1) (X, d) is complete.

(2) Every decreasing sequence $(C_n)_{n \in \mathbb{N}}$ of nonempty closed sets of vanishing diameter, i.e. $\operatorname{diam}(C_n) \to 0$, has a nonempty intersection, i.e. $\bigcap_n C_n \neq \emptyset$.

(3) Every decreasing sequence $(B_n)_{n\in\mathbb{N}}$ of closed balls of vanishing diameter has a nonempty intersection.

Proof. (2) $\Rightarrow$ (3). Trivial.

(1) $\Rightarrow$ (2). Given a decreasing seq. (C_n) of closed sets with $\operatorname{diam}(C_n)\to 0$.



Let $x_n \in C_n$ (this uses Axiom of Choice).
Then $\{x_n, x_{n+1}, \dots\} \subseteq C_n$, so $\operatorname{diam}(\{x_n, x_{n+1}, \dots\})$
$\leq \operatorname{diam}(C_n) \to 0$ as $n \to \infty$, so
(x_n) is Cauchy, hence has a limit $x \in X$.
Since $\{x_n, x_{n+1}, \dots\} \subseteq C_n$ and C_n is closed, it has to contain the limit x.
Thus, each C_n contains x, so $x \in \bigcap_n C_n$. □

(2) $\Rightarrow$ (1). Let (x_n) be a Cauchy sequence.
Let $C_n := \overline{\{x_n, x_{n+1}, \dots\}}$, so C_n is closed and $\operatorname{diam}(C_n) = \operatorname{diam}\{x_n, x_{n+1}, \dots\} \to 0$ as $n \to \infty$.
Thus, $\exists\, x \in \bigcap_n C_n$. $d(x_n, x) \leq \operatorname{diam}(C_n) \to 0$. □

(3) $\Rightarrow$ (1). Let (x_n) be a Cauchy sequence. It's enough to show that it has a convergent subsequence.
Acceleration trick: choose a subsequence (x_{n_k}) s.t. $\operatorname{diam}(x_{n_k}, x_{n_{k+1}}, x_{n_{k+2}}, \dots) \leq 2^{-k}$. (Build

this recursively.) Without loss of generality, we may thus assume that the original (x_n) had this property: $\operatorname{diam}(\{x_n, x_{n+1}, \dots\}) \leq 2^{-n}$.
Let $B_n := \overline{B}_{2\cdot 2^{-n}}(x_n)$. Then $B_{n+1} \subseteq B_n$ because if $x \in B_{n+1}$, then $d(x, x_{n+1}) \leq 2\cdot 2^{-(n+1)}$ and $d(x_n, x_{n+1}) \leq 2^{-n}$, so by the Δ-ineq.
$d(x, x_n) \leq 2\cdot 2^{-(n+1)} + 2^{-n} = 2\cdot 2^{-n}$, so $x \in B_n$. $\square$

Counterexamples for $\operatorname{diam} \not\to 0$.

- Let $X := \mathbb{N}$ with the discrete metric. Then all sets are closed, and we take $C_n := \{n, n+1, n+2, \dots\}$. $\operatorname{diam}(C_n) = 1 \not\to 0$, and $\bigcap_n C_n = \emptyset$.
- Again let $X := \mathbb{N}$ with the following metric
  $d(n,m) := 1 + 2^{-\min(n,m)}$ for $n \neq m$,
  and $d(n,n) := 0$.
  Then $B_n := \overline{B}_{1+2^{-n}}(n) = \{n, n+1, n+2, \dots\}$ and $\operatorname{diam}(B_n) = 1 + 2^{-n} \not\to 0$ and $\bigcap_n B_n = \emptyset$.

Both of these metric space don't have nontrivial Cauchy sequence, hence are complete.